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A NOTE ON MERENTES–RIESZ BOUNDED VARIATION SPACES

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Abstract. In this paper we introduce a function space with some generalization of bounded variation and study some of its properties, like embeddings, decompositions and others.

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1. Introduction

Around two centuries ago C. Jordan introduced the notion of a function of bounded variation and established the relation between these functions and monotonic ones when he was studying convergence of Fourier series. Later on the concept of bounded variation was generalized in various directions by many mathematicians, such has, L. Ambrosio, R. Caccioppoli, L. Cesari, E. Conway, G. Dal Maso, E. de Giorgi, S. Hudjaev, J. Musielak, O. Oleinik, W. Orlicz, F. Riesz, J. Smoller, L. Tonelli, A. Vol'pert, N. Wiener, among many others. It is noteworthy to mention that many of these generalizations where motivated by problems in such areas as calculus of variations, convergence of Fourier series, geometric measure theory, mathematical physics, etc. For many applications of functions of bounded variation in mathematical physics see the monograph [6]. For a thorough exposition regarding bounded variation spaces and related topics, see the recent monograph [1].

In 1992 N. Merentes [4] generalized the concept of bounded p-variation in the sense of Riesz defining the notion of bounded $(p,2)$-variation. We say that a function $f:[a,b]\longrightarrow\mathbb{R}$ has bounded $(p,2)$-variation in $[a,b]$ if the number

$$\begin{aligned}\mathsf{V}^{\mathsf{R}}_{p,2}(f)&=\mathsf{V}^{\mathsf{R}}_{p,2}(f,[a,b])\\&=\sup_{\Pi}\sum_{j=1}^{n-1}\left|\frac{f(b_j)-f(d_j)}{b_j-d_j}-\frac{f(c_j)-f(a_j)}{c_j-a_j}\right|^p\frac{1}{(b_j-a_j)^{p-1}}\end{aligned}$$

is finite, where $p\geq 1$ and the supremum is taken on the set of all block partitions of $[a,b]$.

The set of all bounded $(p,2)$-variation functions is denoted by $\mathsf{RV}_{(p,2)}([a,b])$ which has an algebra structure. Moreover, it was shown that all functions that have bounded $(p,2)$-variation also has bounded second-variation. In this work we generalize this concept to obtain the bounded $(p,2,\alpha)$-variation functions in the sense of Riesz and obtain some characterizations of this new space.

2. PRELIMINARIES

Before introducing the bounded variation space we will need some auxiliary results.

Definition 1. A function $f:[a,b]\longrightarrow\mathbb{R}$ is said to be an α-Lipschitz function if there exists a constant $M>0$ such that

$$|f(x)-f(y)|\le M|\alpha(x)-\alpha(y)|$$

for all $x,y\in(a,b)$ with $x\neq y$. We define the space α-$\mathsf{Lip}[a,b]$ as the space of all α-Lipschitz functions. This space is normable, via the norm

$$\|f\|_{\alpha\text{-}\mathsf{Lip}}:=|f(a)|+\sup_{x\neq y}\frac{|f(x)-f(y)|}{|\alpha(x)-\alpha(y)|}.$$

We now introduce the concepts of α absolutely continuous function and α-derivative.

Definition 2. A function $f:[a,b]\to\mathbb{R}$ is said to be *absolutely continuous with respect to* α if, for every $\varepsilon>0$, there exists some $\delta>0$ such that if $\{(a_j,b_j)\}_{j=1}^{n}$ are disjoint open subintervals of $[a,b]$, then

$$\sum_{j=1}^{n}|\alpha(b_j)-\alpha(a_j)|<\delta \quad \text{implies} \quad \sum_{j=1}^{n}|f(b_j)-f(a_j)|<\varepsilon.$$

All functions in α-$\mathsf{AC}[a,b]$ are bounded and form an algebra of functions under pointwise defined standard operations.

Definition 3. Suppose f and α are real-valued functions defined on the same open interval I and let $x_0\in I$. We say that f *is α-differentiable at* x_0 if the following limit exists

$$\lim_{x\to x_0}\frac{f(x)-f(x_0)}{\alpha(x)-\alpha(x_0)}.$$

If the limit exists we denote its value by $f'_\alpha(x_0)$, which we call the *α-derivative of f at x_0*.

3. $\mathsf{RV}_{(p,2,\alpha)}[a,b]$ IS A NORMED SPACE

We want to recall the so-called *Popoviciu variation* (introduced in 1933 by T. Popoviciu in [5]) for a partition $\Pi = \{a = x_1 < x_2 < \cdots < x_m = b\}$ and a function $f : [a,b] \longrightarrow \mathbb{R}$ is given by

$$\operatorname{Var}_{k,1}(f,\Pi,[a,b]) = \sum_{j=1}^{m-k+1} |f[x_j,\ldots,x_{j+k-1}] - f[x_{j-1},\ldots,x_{j+k-2}]|$$

where $f[\cdot,\ldots,\cdot]$ is defined recursively in the following way:

$$\begin{aligned} f[x_0] &:= f(x_0), \\ f[x_0,x_1] &:= \frac{f[x_1]-f[x_0]}{x_1-x_0} \\ f[x_0,x_1,x_2] &:= \frac{f[x_1,x_2]-f[x_0,x_1]}{x_2-x_0} \\ &\cdots\cdots \\ f[x_0,x_1,\ldots,x_k] &:= \frac{f[x_1,x_2,\ldots,x_k]-f[x_0,x_1,\ldots,x_{k-1}]}{x_k-x_0}. \end{aligned}$$

In the following we will consider a block partition Π of the interval $[a,b]$. It will be taken in the following way

$$\begin{aligned} \Pi = \{a = x_{1,1} < x_{1,2} \leqslant x_{1,3} < x_{1,4} = x_{2,1} < x_{2,2} \leqslant x_{2,3} < x_{2,4} \\ = x_{3,1} < \cdots < x_{n-1,4} = x_{n,1} < x_{n,2} \leqslant x_{n,3} < x_{n,4} = b\}, \end{aligned} \tag{3.1}$$

in place of the regular partition.

Definition 4. Let f be a real-valued function defined on $[a,b]$, Π be a block partition of $[a,b]$ and α be an increasing function. Let

$$\sigma^{\mathsf{R}}_{(p,2,\alpha)}(f,\Pi) = \sum_{j=1}^{n} \frac{\left|f_\alpha[x_{j,4},x_{j,3}] - f_\alpha[x_{j,2},x_{j,1}]\right|^p}{|\alpha(x_{j,4})-\alpha(x_{j,1})|^{p-1}}$$

where

$$f_\alpha[a,b] = \frac{f(b)-f(a)}{\alpha(b)-\alpha(a)}$$

and

$$\mathsf{V}^{\mathsf{R}}_{(p,2,\alpha)}(f,[a,b]) = \mathsf{V}^{\mathsf{R}}_{(p,2,\alpha)}(f) = \sup_{\Pi} \sigma^{\mathsf{R}}_{(p,2,\alpha)}(f,\Pi),$$

where the supremum is taken on all block partitions of $[a,b]$. To the number $\mathsf{V}^{\mathsf{R}}_{(p,2,\alpha)}(f,[a,b])$ we call the *Riesz* $(p,2,\alpha)$*-variation of the function* f *in* $[a,b]$. If $\mathsf{V}^{\mathsf{R}}_{(p,2,\alpha)}(f,[a,b]) < \infty$, then we say that f has *bounded Riesz* $(p,2,\alpha)$*-variation*. The set of all functions is denoted by $\mathsf{RV}_{(p,2,\alpha)}([a,b])$.

Remark 1. When $p = 1$ we observe that $\mathsf{BV}_{(2,\alpha)}[(a,b)] = \mathsf{RV}_{(1,2,\alpha)}([a,b])$.

In the following result we will show that if f has Riesz bounded $(p,2,\alpha)$-variation, then f has bounded α-second variation.

Theorem 1. *We have* $\mathsf{RV}_{(p,2,\alpha)}([a,b]) \hookrightarrow \mathsf{BV}_{(2,\alpha)}[(a,b)]$, *with*

$$\mathsf{V}_{(2,\alpha)}(f,[a,b]) \le (\alpha(b)-\alpha(a))^{\frac{p-1}{p}} \mathsf{V}^{\mathsf{R}}_{(p,2,\alpha)}(f,[a,b]). \tag{3.2}$$

Proof. Let Π be a block partition of type (3.1). Using the Hölder inequality we obtain

$$\begin{aligned}
&\sum_{j=1}^{n} \frac{\left|f_\alpha[x_{j,4},x_{j,3}] - f_\alpha[x_{j,2},x_{j,1}]\right|}{|\alpha(x_{j,4})-\alpha(x_{j,1})|^{\frac{p-1}{p}}} |\alpha(x_{j,4})-\alpha(x_{j,1})|^{\frac{p-1}{p}} \\
&\le \left(\sum_{j=1}^{n} \frac{\left|f_\alpha[x_{j,4},x_{j,3}] - f_\alpha[x_{j,2},x_{j,1}]\right|^p}{|\alpha(x_{j,4})-\alpha(x_{j,1})|^{p-1}}\right)^{\frac{1}{p}} \left(\sum_{j=1}^{n} |\alpha(x_{j,4})-\alpha(x_{j,1})|\right)^{\frac{p-1}{p}} \\
&\le \left(\mathsf{V}^{\mathsf{R}}_{(p,2,\alpha)}(f,[a,b])\right)^{\frac{1}{p}} (\alpha(b)-\alpha(a))^{\frac{p-1}{p}}.
\end{aligned}$$

Since the obtained inequality holds for all block partitions we obtain the desired inequality (3.2). □

Remark 2. We know (see [3]) that if $f \in \mathsf{BV}_{(2,\alpha)}[(a,b)]$, then there exists the right and left α-derivative $f'_{\alpha+}(x_0)$ and $f'_{\alpha-}(x_0)$ on each $x_0 \in (a,b)$ and $f'_{\alpha+}(a)$ and $f'_{\alpha-}(b)$. The last result allow us to conclude that this is also true if $f \in \mathsf{RV}_{(p,2,\alpha)}([a,b])$. In particular, there exists $f'_{\alpha+}(a)$ which we write as $f'_\alpha(a)$.

Lemma 1. *Let* $f : [a,b] \longrightarrow \mathbb{R}$ *be a function such that* $f \in \mathsf{RV}_{(p,2,\alpha)}([a,b])$ *and* $\mathsf{V}^{\mathsf{R}}_{(p,2,\alpha)}(f,[a,b]) = 0$. *Then there exists* $\lambda,\mu \in \mathbb{R}$ *such that* $f(x) = \lambda\alpha(x)+\mu$ *for all* $x \in [a,b]$.

Proof. Since $\mathsf{V}^{\mathsf{R}}_{(p,2,\alpha)}(f,[a,b]) = 0$ we have that $\sigma^{\mathsf{R}}_{(p,2,\alpha)}(f,\Pi) = 0$ for all block partitions Π of $[a,b]$. Let us consider the particular partition given by $\Pi_0 = \{a < x \le x < b\}$,

$$\sigma^{\mathsf{R}}_{(p,2,\alpha)}(f,\Pi_0) = \left|\frac{f(b)-f(x)}{\alpha(b)-\alpha(x)} - \frac{f(x)-f(a)}{\alpha(x)-\alpha(a)}\right|^p \frac{1}{|\alpha(b)-\alpha(a)|^{p-1}} = 0,$$

where

$$\left|\frac{f(b)-f(x)}{\alpha(b)-\alpha(x)} - \frac{f(x)-f(a)}{\alpha(x)-\alpha(a)}\right|^p = 0,$$

Direct calculations show that

$$f(x) = \frac{f(b)-f(a)}{\alpha(b)-\alpha(a)}\alpha(x) + \frac{f(a)\alpha(b)-f(b)\alpha(a)}{\alpha(b)-\alpha(a)}$$

and now taking λ and μ in the following manner

$$\lambda = \frac{f(b)-f(a)}{\alpha(b)-\alpha(a)}, \quad \text{and} \quad \mu = \frac{f(a)\alpha(b)-f(b)\alpha(a)}{\alpha(b)-\alpha(a)}$$

we have the desired result. □

Remark 3. The set $\mathsf{RV}_{(p,2,\alpha)}([a,b])$ can be equipped with a linear space structure considering the operator $f \mapsto \|f \mid \mathsf{RV}_{(p,2,\alpha)}([a,b])\|$ defined in the space $\mathsf{RV}_{(p,2,\alpha)}([a,b])$ given in the following way:

$$\|f \mid \mathsf{RV}_{(p,2,\alpha)}([a,b])\| := |f(a)| + |f'_\alpha(a)| + \left(\mathsf{V}^{\mathsf{R}}_{(p,2,\alpha)}(f,[a,b])\right)^{\frac{1}{p}}, \tag{3.3}$$

for $f \in \mathsf{RV}_{(p,2,\alpha)}([a,b])$.

Theorem 2. *The operator* $f \mapsto \|f \mid \mathsf{RV}_{(p,2,\alpha)}([a,b])\|$ *is a norm in the space* $\mathsf{RV}_{(p,2,\alpha)}([a,b])$.

Proof. Let us take f such that $\|f \mid \mathsf{RV}_{(p,2,\alpha)}([a,b])\| = 0$. From (3.3) this means that $|f(a)| = 0$, $|f'_\alpha(a)| = 0$ and $\mathsf{V}^{\mathsf{R}}_{(p,2,\alpha)}(f) = 0$. Since $\mathsf{V}^{\mathsf{R}}_{(p,2,\alpha)}(f) = 0$ from Lemma 1 we deduce that

$$f(x) = \lambda\alpha(x) + \mu, \quad \lambda,\mu \in \mathbb{R}.$$

Since $f'_\alpha(x) = \lambda$ for $x \in [a,b]$ we conclude that f'_α is a constant function which is null at $x = a$ and thus $\lambda = 0$, from which we get that $f \equiv 0$ since $f(a) = 0$.

It is straighforward to see that for $\lambda \in \mathbb{R}$ we have that

$$\mathsf{V}^{\mathsf{R}}_{(p,2,\alpha)}(\lambda f) = |\lambda|^p \mathsf{V}^{\mathsf{R}}_{(p,2,\alpha)}(f)$$

and now by the definition of the operator (3.3) implies the homogeneity of the operator under consideration.

Let us now prove the triangle inequality. We now introduce the following notation

$$\Delta^j_\alpha(f) = f_\alpha[x_{j,4},x_{j,3}] - f_\alpha[x_{j,2},x_{j,1}].$$

Let $f,g \in \mathsf{RV}_{(p,2,\alpha)}([a,b])$ and Π be a block partition of $[a,b]$ as in (3.1), then

$$\begin{aligned}
\sigma^{\mathsf{R}}_{(p,2,\alpha)}(f+g,\Pi) &= \sum_{j=1}^{n} \frac{|\Delta^j_\alpha(f+g)|^p}{|\alpha(x_{j,4})-\alpha(x_{j,1})|^{p-1}} \\
&= \sum_{j=1}^{n} \frac{|\Delta^j_\alpha(f)+\Delta^j_\alpha(g)|^p}{|\alpha(x_{j,4})-\alpha(x_{j,1})|^{p-1}} \\
&\le \sum_{j=1}^{n} \frac{|\Delta^j_\alpha(f)+\Delta^j_\alpha(g)|^{p-1}}{|\alpha(x_{j,4})-\alpha(x_{j,1})|^{p-1}} |\Delta^j_\alpha(f)|
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=1}^{n} \frac{|\Delta_\alpha^j(f)+\Delta_\alpha^j(g)|^{p-1}}{|\alpha(x_{j,4})-\alpha(x_{j,1})|^{p-1}}|\Delta_\alpha^j(g)| \\
&\le \sum_{j=1}^{n} \frac{|\Delta_\alpha^j(f)+\Delta_\alpha^j(g)|^{p-1}}{|\alpha(x_{j,4})-\alpha(x_{j,1})|^{\frac{(p-1)2}{p}}} \frac{|\Delta_\alpha^j(f)|}{|\alpha(x_{j,4})-\alpha(x_{j,1})|^{\frac{p-1}{p}}} \\
&\quad+\sum_{j=1}^{n} \frac{|\Delta_\alpha^j(f)+\Delta_\alpha^j(g)|^{p-1}}{|\alpha(x_{j,4})-\alpha(x_{j,1})|^{\frac{(p-1)2}{p}}} \frac{|\Delta_\alpha^j(g)|}{|\alpha(x_{j,4})-\alpha(x_{j,1})|^{\frac{p-1}{p}}} \\
&\le \left[\sum_{j=1}^{n} \frac{|\Delta_\alpha^j(f)+\Delta_\alpha^j(g)|^{p}}{|\alpha(x_{j,4})-\alpha(x_{j,1})|^{p-1}}\right]^{\frac{p-1}{p}} \left[\sum_{j=1}^{n} \frac{|\Delta_\alpha^j(f)|^{p}}{|\alpha(x_{j,4})-\alpha(x_{j,1})|^{p-1}}\right]^{\frac{1}{p}} \\
&\quad+\left[\sum_{j=1}^{n} \frac{|\Delta_\alpha^j(f)+\Delta_\alpha^j(g)|^{p}}{|\alpha(x_{j,4})-\alpha(x_{j,1})|^{p-1}}\right]^{\frac{p-1}{p}} \left[\sum_{j=1}^{n} \frac{|\Delta_\alpha^j(g)|^{p}}{|\alpha(x_{j,4})-\alpha(x_{j,1})|^{p-1}}\right]^{\frac{1}{p}}.
\end{aligned}
$$

We therefore have

$$\left(\sigma^{\mathsf{R}}_{(p,2,\alpha)}(f+g,\Pi)\right)^{\frac{1}{p}} \le \left(\sigma^{\mathsf{R}}_{(p,2,\alpha)}(f,\Pi)\right)^{\frac{1}{p}} + \left(\sigma^{\mathsf{R}}_{(p,2,\alpha)}(g,\Pi)\right)^{\frac{1}{p}}$$

and since this is true for all partitions, we have

$$\mathsf{V}^{\mathsf{R}}_{(p,2,\alpha)}(f+g,[a,b]) \le \mathsf{V}^{\mathsf{R}}_{(p,2,\alpha)}(f,[a,b]) + \mathsf{V}^{\mathsf{R}}_{(p,2,\alpha)}(g,[a,b]),$$

and now it easily follows the triangle inequality. □

4. Embedding with $\mathsf{RV}_{(p,2,\alpha)}([a,b])$

We will show that if $p<q$, then there exists an embedding between the spaces $\mathsf{RV}_{(p,2,\alpha)}([a,b])$ and $\mathsf{RV}_{(q,2,\alpha)}([a,b])$. We will need this fact to show completeness of $\mathsf{RV}_{(p,2,\alpha)}([a,b])$.

Theorem 3. *If $1<q<p<\infty$, then $\mathsf{RV}_{(p,2,\alpha)}([a,b]) \hookrightarrow \mathsf{RV}_{(q,2,\alpha)}([a,b])$, with*

$$\left\| f \mid \mathsf{RV}_{(q,2,\alpha)}([a,b])\right\| \le \max\left\{1,(\alpha(b)-\alpha(a))^{\frac{1}{q}-\frac{1}{p}}\right\} \left\| f \mid \mathsf{RV}_{(p,2,\alpha)}([a,b])\right\|,$$

for $f\in \mathsf{RV}_{(p,2,\alpha)}([a,b])$.

Proof. Let $f\in \mathsf{RV}_{(p,2,\alpha)}([a,b])$ and Π be a block partition of $[a,b]$ as in (3.1). Let us consider

$$
\begin{aligned}
\sigma^{\mathsf{R}}_{(q,2,\alpha)}(f,\Pi) &= \sum_{j=1}^{n} \frac{|\Delta_\alpha^j(f)|^q}{|\alpha(x_{j,4})-\alpha(x_{j,1})|^{q-1}} \\
&= \sum_{j=1}^{n} \frac{|\Delta_\alpha^j(f)|^q}{|\alpha(x_{j,4})-\alpha(x_{j,1})|^{\frac{(p-1)q}{p}}} |\alpha(x_{j,4})-\alpha(x_{j,1})|^{\frac{p-q}{p}}.
\end{aligned}
$$

Applying the Hölder inequality we obtain

$$\begin{aligned}\sigma^{\mathsf{R}}_{(q,2,\alpha)}(f,\Pi) &\leq \left[\sum_{j=1}^{n} \frac{|\Delta^{j}_{\alpha}(f)|^{p}}{|\alpha(x_{j,4})-\alpha(x_{j,1})|^{p-1}}\right]^{\frac{q}{p}} \left[\sum_{j=1}^{n} |\alpha(x_{j,4})-\alpha(x_{j,1})|\right]^{\frac{p-q}{p}} \\ &= \left[\sigma^{\mathsf{R}}_{(p,2,\alpha)}(f,\Pi)\right]^{\frac{q}{p}} (\alpha(b)-\alpha(a))^{\frac{p-q}{p}},\end{aligned}$$

from which

$$\left[\mathsf{V}^{\mathsf{R}}_{(q,2,\alpha)}(f,[a,b])\right]^{\frac{1}{q}} \leq (\alpha(b)-\alpha(a))^{\frac{1}{q}-\frac{1}{p}} \left[\mathsf{V}^{\mathsf{R}}_{(p,2,\alpha)}(f,[a,b])\right]^{\frac{1}{p}}$$

whence

$$\left\| f \mid \mathsf{RV}_{(q,2,\alpha)}([a,b])\right\| \leq \max\left\{1,(\alpha(b)-\alpha(a))^{\frac{1}{q}-\frac{1}{p}}\right\} \left\| f \mid \mathsf{RV}_{(p,2,\alpha)}([a,b])\right\|,$$

for $f \in \mathsf{RV}_{(p,2,\alpha)}([a,b])$. □

Remark 4. The proof of the above theorem remains valid if $q = 1$.

Corollary 1. *For $p \geq 1$ we have $\mathsf{RV}_{(p,2,\alpha)}([a,b]) \hookrightarrow \mathsf{BV}_{(2,\alpha)}[(a,b)]$, with*

$$\left\| f \mid \mathsf{BV}_{(2,\alpha)}[(a,b)]\right\| \leq \max\left\{1,(\alpha(b)-\alpha(a))^{\frac{p-1}{p}}\right\} \left\| f \mid \mathsf{RV}_{(p,2,\alpha)}([a,b])\right\|.$$

The following corollary follows from the previous results and from results from [2].

Corollary 2. *If $1 < q < p < \infty$, then*

$$\begin{aligned}&\mathsf{RV}_{(p,2,\alpha)}[a,b] \hookrightarrow \mathsf{RV}_{(q,2,\alpha)}[a,b] \hookrightarrow \mathsf{V}_{(2,\alpha)}[a,b] \hookrightarrow \alpha\text{-}\mathsf{Lip}[a,b] \hookrightarrow \\ &\quad \hookrightarrow \mathsf{RV}_{(p,\alpha)}[a,b] \hookrightarrow \mathsf{RV}_{(q,\alpha)}[a,b] \hookrightarrow \alpha\text{-}\mathsf{AC}[a,b] \hookrightarrow \mathsf{V}[a,b] \hookrightarrow \mathsf{B}[a,b].\end{aligned}$$

5. $\mathsf{RV}_{(p,2,\alpha)}([a,b])$ IS A BANACH SPACE

Theorem 4. *The space $\left(\mathsf{RV}_{(p,2,\alpha)}([a,b]), \left\|\cdot \mid \mathsf{RV}_{(p,2,\alpha)}([a,b])\right\|\right)$ is a Banach space.*

Proof. Let $(f_k)_{k\in\mathbb{N}}$ be a Cauchy sequence in $\mathsf{RV}_{(p,2,\alpha)}([a,b])$. Given $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ such that

$$\left\| f_q - f_r \mid \mathsf{RV}_{(p,2,\alpha)}([a,b])\right\| < \varepsilon$$

if $q,r > N_\varepsilon$, from which we get the following system of inequalities

$$\begin{cases} |f_q(a) - f_r(a)| < \varepsilon, \\ \\ |(f_q)'_\alpha(a) - (f_r)'_\alpha(a)| < \varepsilon, \\ \\ \mathsf{V}^{\mathsf{R}}_{(p,2,\alpha)}(f_q - f_p) < \varepsilon^p. \end{cases}$$

From Corollary 2 we conclude that $(f_k)_{k\in\mathbb{N}}$ is a Cauchy sequence in α-$\mathsf{Lip}[a,b]$, thus we have

$$\left\| f_q - f_r \mid \alpha\text{-}\mathsf{Lip}[a,b] \right\| < \varepsilon K$$

with $K = \max\left\{1, (\alpha(b)-\alpha(a))^{\frac{p-1}{p}}\right\}$, hence

$$|f_q(a) - f_r(a)| + \sup_{x\neq y}\left|\frac{(f_q - f_r)(x) - (f_q - f_r)(y)}{\alpha(x)-\alpha(y)}\right| < \varepsilon K$$

and so

$$|f_q(x) - f_r(x)| < K\varepsilon(1+\alpha(b)-\alpha(a)), \quad x \in [a,b].$$

This tell us that for each $x \in [a,b]$ $(f_k(x))_{k\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$. Since $\mathbb{R}$ is a complete space, we can define $f : [a,b] \longrightarrow \mathbb{R}$ as $x \mapsto f(x) := \lim_{k\to\infty} f_k(x)$. We are about to prove that:

(i) $f \in \mathsf{RV}_{(p,2,\alpha)}([a,b])$, and

(ii) $(f_k)_{k\in\mathbb{N}}$ converges to f in the $\mathsf{RV}_{(p,2,\alpha)}([a,b])$-norm.

(i) Let Π be a block partition of $[a,b]$ as in (3.1) Since $(f_k)_{k\in\mathbb{N}}$ is a Cauchy sequence in $\mathsf{RV}_{(p,2,\alpha)}([a,b])$ the norm sequence $\left(\left\| f_k \mid \mathsf{RV}_{(p,2,\alpha)}([a,b])\right\|\right)_{k\in\mathbb{N}}$ is bounded, that is, there exists $M > 0$ such that $\left\| f \mid \mathsf{RV}_{(p,2,\alpha)}([a,b])\right\| \le M$ for $k \in \mathbb{N}$. From this fact we have

$$\begin{aligned}
&\left[\sum_{j=1}^{n}\left|\frac{f(x_{j,4})-f(x_{j,3})}{\alpha(x_{j,4})-\alpha(x_{j,3})} - \frac{f(x_{j,2})-f(x_{j,1})}{\alpha(x_{j,2})-\alpha(x_{j,1})}\right|^p \frac{1}{|\alpha(x_{j,4})-\alpha(x_{j,1})|^{p-1}}\right]^{\frac{1}{p}} \\
&= \lim_{k\to\infty}\left[\sum_{j=1}^{n}\left|\frac{f_k(x_{j,4})-f_k(x_{j,3})}{\alpha(x_{j,4})-\alpha(x_{j,3})} - \frac{f_k(x_{j,2})-f_k(x_{j,1})}{\alpha(x_{j,2})-\alpha(x_{j,1})}\right|^p \frac{1}{|\alpha(x_{j,4})-\alpha(x_{j,1})|^{p-1}}\right]^{\frac{1}{p}} \\
&\le \lim_{k\to\infty}\left[\mathsf{V}^{\mathsf{R}}_{(p,2,\alpha)}(f_k,[a,b])\right]^{\frac{1}{p}} \\
&\le \lim_{k\to\infty}\left\| f_k \mid \mathsf{RV}_{(p,2,\alpha)}([a,b])\right\| \\
&\le M
\end{aligned}$$

for all partitions Π, then $\left(\mathsf{V}^{\mathsf{R}}_{(p,2,\alpha)}(f)\right)^{\frac{1}{p}} \le M$ and thus $f \in \mathsf{RV}_{(p,2,\alpha)}([a,b])$. Using embedding we may observe that $f'_\alpha(a)$ exists.

(ii) One more time let us consider Π to be a block partition of $[a,b]$ as in (3.1). Let $q,r > N_\varepsilon$, then

$$\sum_{j=1}^{n}\left|\frac{(f_q-f_r)(x_{j,4})-(f_q-f_r)(x_{j,3})}{\alpha(x_{j,4})-\alpha(x_{j,3})}-\frac{(f_q-f_r)(x_{j,2})-(f_q-f_r)(x_{j,1})}{\alpha(x_{j,2})-\alpha(x_{j,1})}\right|^p \times\frac{1}{|\alpha(x_{j,4})-\alpha(x_{j,1})|^{p-1}} \leq \mathsf{V}^{\mathsf{R}}_{(p,2,\alpha)}((f_q-f_r),[a,b]) < \varepsilon^p.$$

Letting $r\to\infty$ the above expression becomes

$$\sum_{j=1}^{n}\left|\frac{(f_q-f)(x_{j,4})-(f_q-f)(x_{j,3})}{\alpha(x_{j,4})-\alpha(x_{j,3})}-\frac{(f_q-f)(x_{j,2})-(f_q-f)(x_{j,1})}{\alpha(x_{j,2})-\alpha(x_{j,1})}\right|^p \times\frac{1}{|\alpha(x_{j,4})-\alpha(x_{j,1})|^{p-1}} < \varepsilon^p.$$

This holds for any partition of $[a,b]$, therefore $\mathsf{V}^{\mathsf{R}}_{(p,2,\alpha)}(f_q-f) < \varepsilon^p$ when $q > N_\varepsilon$. Let $h\in\mathbb{R}^+$ be such that $a<a+h<s<t\leq b$, then

$$\left|\frac{(f_q-f)(t)-(f_q-f)(s)}{\alpha(t)-\alpha(s)}-\frac{(f_q-f)(a+h)-(f_q-f)(a)}{\alpha(a+h)-\alpha(a)}\right|^p\frac{1}{|\alpha(t)-\alpha(s)|^{p-1}} \leq \mathsf{V}^{\mathsf{R}}_{(p,2,\alpha)}(f_q-f) < \varepsilon^p$$

if $q > N_\varepsilon$. Letting $h\to 0$ we have

$$\left|\frac{(f_q-f)(t)-(f_q-f)(s)}{\alpha(t)-\alpha(s)}-(f_q-f)'_\alpha(a)\right|^p\frac{1}{|\alpha(t)-\alpha(s)|^{p-1}} \leq \varepsilon^p$$

from which

$$\left|\frac{(f_q-f)(t)-(f_q-f)(s)}{\alpha(t)-\alpha(s)}-(f_q)'_\alpha(a)+(f)'_\alpha(a)\right|^p \leq \varepsilon^p|\alpha(t)-\alpha(s)|^{p-1}.$$

Since $f_q, f\in \mathsf{RV}_{(p,2,\alpha)}([a,b])\hookrightarrow \alpha\text{-}\mathsf{Lip}[a,b]$, we have

$$\begin{aligned}|(f_q)'_\alpha(a)-(f)'_\alpha(a)| &\leq \varepsilon|\alpha(b)-\alpha(a)|^{\frac{p-1}{p}}+\left|\frac{(f_q-f)(t)-(f_q-f)(s)}{\alpha(t)-\alpha(s)}\right| \\ &\leq \varepsilon|\alpha(b)-\alpha(a)|^{\frac{p-1}{p}}+\left\|f_q-f \mid \alpha\text{-}\mathsf{Lip}[a,b]\right\| \\ &\leq \varepsilon|\alpha(b)-\alpha(a)|^{\frac{p-1}{p}}+\varepsilon K=\tilde{K}\varepsilon\end{aligned}$$

if $q > N_\varepsilon$, where

$$\tilde{K}=\max\left\{K,(\alpha(b)-\alpha(a))^{\frac{p-1}{p}}\right\}$$

since $\| f_q - f \mid \alpha\text{-Lip}[a,b] \| = \lim_{r\to\infty} \| f_q - f_r \mid \alpha\text{-Lip}[a,b] \| < K\varepsilon$. Finally, for $q > N_\varepsilon$ we obtain

$$|f_q(a) - f(a)| + |(f_q)'_\alpha(a) - (f)'_\alpha(a)| + \left[\mathsf{V}^{\mathsf{R}}_{(p,2,\alpha)}(f_q - f,[a,b])\right]^{\frac{1}{q}} \le (\tilde{K}+2)\varepsilon,$$

in other words

$$\| f_q - f \mid \mathsf{RV}_{(p,2,\alpha)}([a,b]) \| \le (\tilde{K}+2)\varepsilon$$

if $q > N_\varepsilon$ which means that $(f_k)_{k\in\mathbb{N}}$ converges to $f \in \mathsf{RV}_{(p,2,\alpha)}([a,b])$ in the norm $\| \cdot \mid \mathsf{RV}_{(p,2,\alpha)}([a,b]) \|$. □

6. $\mathsf{RV}_{(p,2,\alpha)}([a,b])$ is a Banach Algebra

We are going to show that $\mathsf{RV}_{(p,2,\alpha)}([a,b])$ is closed under the multiplication of functions.

Theorem 5. *Let $f,g \in \mathsf{RV}_{(p,2,\alpha)}([a,b])$. Then $fg \in \mathsf{RV}_{(p,2,\alpha)}([a,b])$.*

Proof. Let Π be a block partition of $[a,b]$ as in (3.1). Let us consider

$$\begin{aligned}\sigma^{\mathsf{R}}_{(p,2,\alpha)}(f\cdot g,\Pi) &= \sum_{j=1}^{n} \frac{\left|(fg)_\alpha[x_{j,4},x_{j,3}] - (fg)_\alpha[x_{j,2},x_{j,1}]\right|^p}{|\alpha(x_{j,4})-\alpha(x_{j,1})|^{p-1}} \\ &= \sum_{j=1}^{n} \frac{\left|(fg)_\alpha[x_{j,4},x_{j,3}] - (fg)_\alpha[x_{j,2},x_{j,1}]\right|^{p-1}}{|\alpha(x_{j,4})-\alpha(x_{j,1})|^{p-1}} \\ &\qquad \times \left|(fg)_\alpha[x_{j,4},x_{j,3}] - (fg)_\alpha[x_{j,2},x_{j,1}]\right|. \end{aligned} \tag{6.1}$$

Observe that the term $\left|(fg)_\alpha[x_{j,4},x_{j,3}] - (fg)_\alpha[x_{j,2},x_{j,1}]\right| =: T$ can be written as

$$\begin{aligned} T = |&f(x_{j,4})g_\alpha[x_{j,4},x_{j,3}] + g(x_{j,3})f_\alpha[x_{j,4},x_{j,3}] \\ &- f(x_{j,2})g_\alpha[x_{j,2},x_{j,1}] - g(x_{j,1})f_\alpha[x_{j,2},x_{j,1}]| \end{aligned}$$

and now adding and subtracting appropriate terms and grouping the terms we obtain

$$\begin{aligned} T = |&f(x_{j,4})(g_\alpha[x_{j,4},x_{j,3}] - g_\alpha[x_{j,2},x_{j,1}]) + g(x_{j,3})(f_\alpha[x_{j,4},x_{j,3}] - f_\alpha[x_{j,2},x_{j,1}]) \\ &+ (f(x_{j,4}) - f(x_{j,2}))g_\alpha[x_{j,2},x_{j,1}] + (g(x_{j,3}) - g(x_{j,1}))f_\alpha[x_{j,2},x_{j,1}]|. \end{aligned} \tag{6.2}$$

From Corollary 2 we have

$$\mathsf{RV}_{(p,2,\alpha)}([a,b]) \hookrightarrow \alpha\text{-Lip}[a,b] \quad \text{and} \quad \mathsf{RV}_{(p,2,\alpha)}([a,b]) \hookrightarrow \mathsf{B}[a,b].$$

This let us write

$$|f(x_{j,4})| \le \|f\|_\infty, \quad |g(x_{j,3})| \le \|g\|_\infty$$

and

$$\frac{f(\xi)-f(\eta)}{\alpha(\xi)-\alpha(\eta)} \le \mathsf{Lip}_\alpha(f), \quad \frac{g(\xi)-g(\eta)}{\alpha(\xi)-\alpha(\eta)} \le \mathsf{Lip}_\alpha(g),$$

for all $\xi,\eta\in[a,b]$. Using these estimates we obtain

$$\begin{aligned}
&|f(x_{j,4})-f(x_{j,2})||g_\alpha[x_{j,2},x_{j,1}]|+|g(x_{j,3}-g(x_{j,1})||f_\alpha[x_{j,2},x_{j,1}]|\\
&=\frac{|f(x_{j,4})-f(x_{j,2})|}{|\alpha(x_{j,4})-\alpha(x_{j,2})|}\cdot\frac{|g(x_{j,2})-g(x_{j,1})|}{|\alpha(x_{j,2})-\alpha(x_{j,1})|}||\alpha(x_{j,4})-\alpha(x_{j,1})||\\
&\quad+\frac{|f(x_{j,2})-f(x_{j,1})|}{|\alpha(x_{j,2})-\alpha(x_{j,1})|}\cdot\frac{|g(x_{j,3})-g(x_{j,1})|}{|\alpha(x_{j,3})-\alpha(x_{j,1})|}|\alpha(x_{j,4})-\alpha(x_{j,1})|\\
&\le(\mathsf{Lip}_\alpha f)(\mathsf{Lip}_\alpha g)\left(\alpha(x_{j,4}-\alpha(x_{j,2})+\alpha(x_{j,3})-\alpha(x_{j,1}))\right)
\end{aligned}$$

replacing all this into (6.2) we have

$$\begin{aligned}
&\left|(fg)_\alpha[x_{j,4},x_{j,3}]-(fg)_\alpha[x_{j,2},x_{j,1}]\right|\\
&\le\|f\|_\infty\left|g_\alpha[x_{j,4},x_{j,3}]-g_\alpha[x_{j,2},x_{j,1}]\right|+\|g\|_\infty\left|f_\alpha[x_{j,4},x_{j,3}]-f_\alpha[x_{j,2},x_{j,1}]\right|\\
&\quad+(\mathsf{Lip}_\alpha f)(\mathsf{Lip}_\alpha g)\left(\alpha(x_{j,4})-\alpha(x_{j,2})+\alpha(x_{j,3})-\alpha(x_{j,1}))\right).
\end{aligned}$$

Now replacing this last estimate into (6.1), separating summations and fixing exponents to apply the Hölder inequality we have

$$\begin{aligned}
\sigma^{\mathsf{R}}_{(p,2,\alpha)}(fg,\Pi)\le\|f\|_\infty&\sum_{j=1}^{n}\frac{\left|(fg)_\alpha[x_{j,4},x_{j,3}]-(fg)_\alpha[x_{j,2},x_{j,1}]\right|^{p-1}}{|\alpha(x_{j,4})-\alpha(x_{j,1})|^{\frac{(p-1)^2}{p}}}\\
&\times\frac{\left|g_\alpha[x_{j,4},x_{j,3}]-g_\alpha[x_{j,2},x_{j,1}]\right|}{|\alpha(x_{j,4})-\alpha(x_{j,1})|^{\frac{p-1}{p}}}\\
&+\|g\|_\infty\sum_{j=1}^{n}\frac{\left|(fg)_\alpha[x_{j,4},x_{j,3}]-(fg)_\alpha[x_{j,2},x_{j,1}]\right|^{p-1}}{|\alpha(x_{j,4})-\alpha(x_{j,1})|^{\frac{(p-1)^2}{p}}}\\
&\times\frac{\left|f_\alpha[x_{j,4},x_{j,3}]-f_\alpha[x_{j,2},x_{j,1}]\right|}{|\alpha(x_{j,4})-\alpha(x_{j,1})|^{\frac{p-1}{p}}}\\
&+(\mathsf{Lip}_\alpha f)(\mathsf{Lip}_\alpha g)\sum_{j=1}^{n}\frac{\left|(fg)_\alpha[x_{j,4},x_{j,3}]-(fg)_\alpha[x_{j,2},x_{j,1}]\right|^{p-1}}{|\alpha(x_{j,4})-\alpha(x_{j,1})|^{\frac{(p-1)^2}{p}}}\\
&\times\frac{|\alpha(x_{j,4})-\alpha(x_{j,2})+\alpha(x_{j,3})-\alpha(x_{j,1})|}{|\alpha(x_{j,4})-\alpha(x_{j,1})|^{\frac{p-1}{p}}}.
\end{aligned}$$

Applying in each summation the Hölder inequality we obtain

$$\begin{aligned}
\sigma^{\mathsf{R}}_{(p,2,\alpha)}(fg,\Pi)\le\|f\|_\infty&\left[\sigma^{\mathsf{R}}_{(p,2,\alpha)}(fg,\Pi)\right]^{\frac{p-1}{p}}\left[\sigma^{\mathsf{R}}_{(p,2,\alpha)}(g,\Pi)\right]^{\frac{1}{p}}\\
&+\|g\|_\infty\left[\sigma^{\mathsf{R}}_{(p,2,\alpha)}(fg,\Pi)\right]^{\frac{p-1}{p}}\left[\sigma^{\mathsf{R}}_{(p,2,\alpha)}(f,\Pi)\right]^{\frac{1}{p}}
\end{aligned}$$

$$+(\mathsf{Lip}_\alpha f)(\mathsf{Lip}_\alpha g)\left[\sigma^{\mathsf{R}}_{(p,2,\alpha)}(fg,\Pi)\right]^{\frac{p-1}{p}}$$
$$\times\left[\sum_{j=1}^{n}\frac{|\alpha(x_{j,4})-\alpha(x_{j,2})+\alpha(x_{j,3})-\alpha(x_{j,1})|^p}{|\alpha(x_{j,4})-\alpha(x_{j,1})|^{p-1}}\right]^{\frac{1}{p}}.$$

Simplifying we have

$$\left[\sigma^{\mathsf{R}}_{(p,2,\alpha)}(fg,\Pi)\right]^{\frac{1}{p}}\leq\|f\|_\infty\left[\sigma^{\mathsf{R}}_{(p,2,\alpha)}(g,\Pi)\right]^{\frac{1}{p}}+\|g\|_\infty\left[\sigma^{\mathsf{R}}_{(p,2,\alpha)}(f,\Pi)\right]^{\frac{1}{p}}$$
$$+(\mathsf{Lip}_\alpha f)(\mathsf{Lip}_\alpha g)\left[\sum_{j=1}^{n}\frac{|\alpha(x_{j,4})-\alpha(x_{j,2})+\alpha(x_{j,3})-\alpha(x_{j,1})|^p}{|\alpha(x_{j,4})-\alpha(x_{j,1})|^{p-1}}\right]^{\frac{1}{p}} \tag{6.3}$$

Simplifying the last bracket in (6.3), for $j=1,\dots,n$, we can observe that

$$\frac{|\alpha(x_{j,4})-\alpha(x_{j,2})+\alpha(x_{j,3})-\alpha(x_{j,1})|^p}{|\alpha(x_{j,4})-\alpha(x_{j,1})|^{p-1}}$$
$$=\left|\frac{\alpha(x_{j,4})-\alpha(x_{j,2})+\alpha(x_{j,3})-\alpha(x_{j,1})}{\alpha(x_{j,4})-\alpha(x_{j,1})}\right|^{p-1}|\alpha(x_{j,4})-\alpha(x_{j,2})+\alpha(x_{j,3})-\alpha(x_{j,1})|$$
$$=\left|1+\frac{\alpha(x_{j,3})-\alpha(x_{j,2})}{\alpha(x_{j,4})-\alpha(x_{j,1})}\right|^{p-1}|\alpha(x_{j,4})-\alpha(x_{j,2})+\alpha(x_{j,3})-\alpha(x_{j,1})|. \tag{6.4}$$

Since $x_{j,1}<x_{j,2}\leq x_{j,3}<x_{j,4}$ we have

$$\alpha(x_{j,3})-\alpha(x_{j,2})<\alpha(x_{j,4})-\alpha(x_{j,1});$$
$$\alpha(x_{j,4})-\alpha(x_{j,2})>0;$$
$$\alpha(x_{j,3})-\alpha(x_{j,1})>0.$$

Substituting this into (6.4)

$$\frac{|\alpha(x_{j,4})-\alpha(x_{j,2})+\alpha(x_{j,3})-\alpha(x_{j,1})|^p}{|\alpha(x_{j,4})-\alpha(x_{j,1})|^{p-1}}$$
$$\leq 2^{p-1}\left(\alpha(x_{j,4})-\alpha(x_{j,2})+\alpha(x_{j,3})-\alpha(x_{j,1})\right)$$

and now summing we obtain

$$\sum_{j=1}^{n}\frac{|\alpha(x_{j,4})-\alpha(x_{j,2})+\alpha(x_{j,3})-\alpha(x_{j,1})|^p}{|\alpha(x_{j,4})-\alpha(x_{j,1})|^{p-1}}\leq 2^p(\alpha(b)-\alpha(a)).$$

Substituting this inequality into (6.3) we have

$$\left[\sigma^{\mathsf{R}}_{(p,2,\alpha)}(fg,\Pi)\right]^{\frac{1}{p}}\leq\|f\|_\infty\left[\mathsf{V}^{\mathsf{R}}_{(p,2,\alpha)}(g,[a,b])\right]^{\frac{1}{p}}+\|g\|_\infty\left[\mathsf{V}^{\mathsf{R}}_{(p,2,\alpha)}(f,[a,b])\right]^{\frac{1}{p}}$$
$$+2(\alpha(b)-\alpha(a))^{\frac{1}{p}}(\mathsf{Lip}_\alpha f)(\mathsf{Lip}_\alpha g),$$

this last inequality holds for all partition Π of $[a,b]$, then

$$\begin{aligned}\left[\mathsf{V}^{\mathsf{R}}_{(p,2,\alpha)}(fg,[a,b])\right]^{\frac{1}{p}} &\le \|f\|_\infty \left[\mathsf{V}^{\mathsf{R}}_{(p,2,\alpha)}(g,[a,b])\right]^{\frac{1}{p}} \\ &\quad + \|g\|_\infty \left[\mathsf{V}^{\mathsf{R}}_{(p,2,\alpha)}(f,[a,b])\right]^{\frac{1}{p}} \\ &\quad + 2(\alpha(b)-\alpha(a))^{\frac{1}{p}}(\mathsf{Lip}_\alpha f)(\mathsf{Lip}_\alpha g) < \infty, \end{aligned} \tag{6.5}$$

from which it follows that $fg \in \mathsf{RV}_{(p,2,\alpha)}([a,b])$. □

We now obtain the following corollary.

Corollary 3. *Let $f,g \in \mathsf{RV}_{(p,2,\alpha)}([a,b])$, then*

$$\begin{aligned}\left\| fg \mid \mathsf{RV}_{(p,2,\alpha)}([a,b])\right\| &\le P \left\| f \mid \mathsf{RV}_{(p,2,\alpha)}([a,b])\right\| \left\| g \mid \mathsf{RV}_{(p,2,\alpha)}([a,b])\right\| \\ &\quad + \|f\|_\infty \left\| g \mid \mathsf{RV}_{(p,2,\alpha)}([a,b])\right\| \\ &\quad + \|g\|_\infty \left\| f \mid \mathsf{RV}_{(p,2,\alpha)}([a,b])\right\|. \end{aligned}$$

with $P = 2\max\left\{(\alpha(b)-\alpha(a))^{\frac{1}{p}}, (\alpha(b)-\alpha(a))^{\frac{2p-2}{p}}\right\}$.

Proof. Note that

$$\begin{aligned}\mathsf{Lip}_\alpha(f) \le \|f \mid \alpha\text{-}\mathsf{Lip}[a,b]\| &\le \left\| f \mid \mathsf{V}_{(2,\alpha)}[a,b]\right\| \\ &\le \max\left\{1, (\alpha(b)-\alpha(a))^{\frac{p-1}{p}}\right\} \left\| f \mid \mathsf{RV}_{(p,2,\alpha)}([a,b])\right\| \end{aligned}$$

and similarly to g. Then (6.5) can be written as

$$\begin{aligned}\left[\mathsf{V}^{\mathsf{R}}_{(p,2,\alpha)}(fg,[a,b])\right]^{\frac{1}{p}} &\le \|f\|_\infty \left[\mathsf{V}^{\mathsf{R}}_{(p,2,\alpha)}(g,[a,b])\right]^{\frac{1}{p}} + \|g\|_\infty \left[\mathsf{V}^{\mathsf{R}}_{(p,2,\alpha)}(f,[a,b])\right]^{\frac{1}{p}} \\ &\quad + 2\,(\alpha(b)-\alpha(a))^{\frac{1}{p}} \left[\max\left\{1, (\alpha(b)-\alpha(a))^{\frac{p-1}{p}}\right\}\right]^2 \\ &\quad \times \left\| f \mid \mathsf{RV}_{(p,2,\alpha)}([a,b])\right\| \left\| g \mid \mathsf{RV}_{(p,2,\alpha)}([a,b])\right\|. \end{aligned}$$

□

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